JOURNAL OF APPROXIMATION THEORY 25, 189–192 (1979)

Efficient Co-Monotone Approximation

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Communicated by Oved Shisha

Received May 11, 1977

The problem of approximating a piecewise monotonic function by a polynomial with the same monotonicity has been considered by several authors (see, e.g., [1–4]). In [2] it is asked whether such "co-monotone" approximation can always be done by nth degree polynomials with an error of $A\omega(f; 1/n)$. (This is the "efficient co-monotone" approximation we refer to in the title). Passow and Raymon [1] have obtained some partial results (for example, they obtain the proximity $A_\epsilon\omega(f; (1/n)(1-\epsilon))$ for every $\epsilon > 0$) but the question has apparently not yet been answered and this is the purpose of this note.

We give the affirmative answer by making what seems to be the most obvious construction for this required approximator. What is not obvious, perhaps, is the requisite error bound which shows that this really does the job.

The construction can be symbolically viewed as $\int \rho \cdot (f'/\rho * K)$, where f is the given function (properly modified), ρ is the polynomial with zeros precisely at the "turning points" of f, K is the Jackson kernel, and $\int$ represents the integral operator. Since K is a positive kernel it is clear that this constructed polynomial is co-monotone with ρ, and hence with f, and we turn to the details and estimations.

First of all the simplest setting is the circle rather than the interval and so we work there.

Next we define the "property modified" $\bar{f}(x)$. To begin with we break our interval into equal pieces of length (around) $1/n$ and we set $g(x)$ equal to the step function which, in the subinterval (a, b), is equal to $[f(b)-f(a)]/(b-a)$. Next we set $h(x) = 0$ in every subinterval, and its adjacent subintervals, wherein f has a turning point, $h(x) = g(x)$ otherwise. Then we set $j(x) = (1+\delta)\, h(x)$, where $h \geqslant 0$, $j(x) = h(x)$ otherwise, δ chosen so that $\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \rho(u)[j(u+t)/\rho(u+t)]\, K(t)\, dt\, du = 0$.

* Supported in part by NSF Grant No. 550-34901.

0021-9045/79/030189-04$02.00/0

Elementary estimates show that this $\delta \ll C\omega(f; 1/n)$. We finally define $\bar{f}$ by $\bar{f}(0) = f(0)$ and $\bar{f}'(x) = j(x)$.

The properties we need to record for this function are the following:

$$\bar{f} - f \ll C\omega(f; 1/n), \tag{1}$$

$$0 \leqslant \bar{f}'/\rho \leqslant Cn^2\omega(f; 1/n), \tag{2}$$

$$\omega(\bar{f}; 1/n) \leqslant C\omega(f; 1/n), \tag{3}$$

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \rho(u)\,[\bar{f}'(u+t)/\rho(u+t)]\,K(t)\,dt\,du = 0. \tag{4}$$

The property (2) follows from the fact that $\bar{f}' \ll n\omega(f; 1/n)$ while at every point where $\bar{f}' \neq 0$ we are at distance $\geqslant 1/n$ from any zero of ρ so that $|\rho| \geqslant c \cdot 1/n$. Property (4) is of vital importance. It insures that our constructed $\int \rho \cdot (\bar{f}'/\rho) * K)$ is a *polynomial*!

We also record the basic properties of the Jackson kernel, $K(t)$. •

$$K(t) \text{ is an } n\text{th degree cosine polynomial,} \tag{5}$$

$$K(t) \geqslant 0, \qquad \int_{-\pi}^{\pi} K(t)\,dt = 1, \qquad \int_{-\pi}^{\pi} t^2K(t)\,dt \leqslant \frac{C}{n^2}. \tag{6}$$

For any $F(x)$ of period 2π,

$$\int_{-\pi}^{\pi} F(x+t)\,K(t)\,dt - F(x) \ll C\omega\left(F; \frac{1}{n}\right). \tag{7}$$

Actually property (7) is an easy consequence of property (6) but we list it separately for reference purposes.

We may assume w.l.o.g. that $f(0) = 0$, so that $\bar{f}(0) = 0$. Our constructed polynomial is then

$$P(x) = \int_0^x \int_{-\pi}^{\pi} \rho(u) \frac{\bar{f}'(u+t)}{\rho(u+t)} K(t)\,dt\,du. \tag{8}$$

On the other hand

$$\begin{aligned}
&\int_0^x \int_{-\pi}^{\pi} \rho(u+t) \frac{\bar{f}'(u+t)}{\rho(u+t)} K(t)\,dt\,du \\
&\quad = \int_0^x \int_{-\pi}^{\pi} \bar{f}'(u+t)\,K(t)\,dt\,du = \int_{-\pi}^{\pi} (\bar{f}(x+t) - \bar{f}(t))\,K(t)\,dt \\
&\quad = \bar{f}(x) - \bar{f}(0) + O\left[\omega\left(\bar{f}, \frac{1}{n}\right)\right] \qquad \text{(by (7))} \\
&\quad = \bar{f}(x) + O\left[\omega\left(\bar{f} \frac{1}{n}\right)\right].
\end{aligned} \tag{9}$$

Thus we need only estimate the difference

$$S = \int_0^x \int_{-\pi}^{\pi} (\rho(u+t) - \rho(u)) \frac{\bar{f}'(u+t)}{\rho(u+t)} K(t)\, dt\, du \tag{10}$$

Now we invert the order of integration and obtain $S = \int_{-\pi}^{\pi} G(t)\, K(t)\, dt$, where

$$G(t) = \int_0^x (\rho(u+t) - \rho(u)) \frac{\bar{f}'(u+r)}{\rho(u+t)}\, du. \tag{11}$$

And, since $K(t)$ is even by (5), we have

$$S = \frac{1}{2} \int_{-\pi}^{\pi} [G(t) + G(-t)]\, K(t)\, dt. \tag{12}$$

We must show that $S = O[\omega(\bar{f}, 1/n)]$ and, by (6), it therefore suffices to show that

$$G(t) + G(-t) = O[n^2 \omega(\bar{f}, 1/n)]\, t^2. \tag{13}$$

We show quite generally that if $g(t) = \int_0^x (\rho(u+t) - \rho(u))\, r(u+t)\, du$, where $|\, r(u+t)| \leqslant M$, then

$$g(t) + g(-t) \leqslant CMt^2 \tag{14}$$

And our required inequality will thereby follow from (2).

To prove (14) we write

$$g(t) + g(-t) = \int_0^x (r(u+t) - r(u-t))\, (\rho(u+t) - \rho(u))\, du$$
$$+ \int_0^x r(u-t)(\rho(u+t) - 2\rho(u) + \rho(u-t))\, du. \tag{15}$$

Since $\rho(u+t) - 2\rho(u) + \rho(u-t) = t^2 \rho''(\xi)$ by the mean value theorem for second differences we see that the second term in (15) is estimated by $|\, x\,| \cdot M \cdot \operatorname{Max} |\, \rho''\,| \cdot t^2 \leqslant CMt^2$, where $C = \pi \operatorname{Max} |\, \rho''\,|$. As for the first term, we integrate it by parts and thereby write it as

$$\left[\int_{u-t}^{u+t} r(v)\, dv\right] (\rho(u+t) - \rho(u-t))|_0^x$$
$$- \int_0^x \left[\int_{u-t}^{u+t} r(v)\, dv\right] (\rho'(u+t) - \rho'(u-t)\, du$$

and this, in turn, is estimated by $2tM \cdot 2t \operatorname{Max} |\, \rho'\,| + \pi \cdot 2tM \cdot 2t \operatorname{Max} |\, \rho''\,| \leqslant CMt^2$, as required. The proof is complete.

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